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Spin-flip cross section from variational calculus: an application to π^+–p scattering

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Abstract. Numerical upper bounds are found for the total spin-flip cross section when the total cross section and the forward slope are known and the partial waves are unitary.

1. Introduction

In two previous articles (Sakmar 1981a, b) the method of application of variational calculus to scattering problems in particle physics (McDowell and Martin 1964, Einhorn and Blankenbecler 1971) was generalised to spin cases. In this paper we apply the method numerically to π^+–p scattering. The application is intended as a demonstration of a rigorous method which takes the spin and unitarity fully into account. The bounds obtained should not be compared with experimental values but rather with what the maximised quantity would be without the variational calculus. The example at hand is the problem of maximising the spin-flip cross section, that is the angle integral of the square of the spin-flip amplitude, with the forward slope and total cross section as constraints. The unitarity of the partial waves gives additional inequality constraints. We had found the form of the partial waves in each of the four classes, defined according to the elasticity of the pair of amplitudes with the same l value (l_+ and l_-). They were: both elastic; both inelastic; f_{l+} elastic, f_{l-} inelastic and f_{l+} inelastic, f_{l-} elastic.

The problem now is to choose partial waves from different classes to fit the constraints and maximise the spin-flip cross section. In § 2 we define and give the basic quantities to be used in the calculation. In § 3 we construct from the existing phase shifts at a large number of energies the total cross section and the forward slope. In § 4 we search for solutions using guidelines from the magnitudes of the constraints and conditions resulting from the variational method. Even though not all classes contribute to the maximum, we considered all of them to keep the approach as general as possible. Thus in addition to maxima saddle points were also found. Those saddle points can be called minimal, in the sense that only those second derivatives of the Lagrange function are positive over which we have no control. All

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other second derivatives are made negative by choosing the Lagrange parameters properly. In § 5 we discuss our results and draw conclusions for further applications.

2. Basic formulae

The quantity to be maximised is the spin-flip cross section σ_{SF} with σ^T the total cross section and $dA/dt|_{t=0}$ as constraints. A is the imaginary part of the scattering amplitude. To simplify the calculations we defined instead

$$G=\frac{k^2}{4\pi}\sigma_{SF}=\sum\frac{l(l+1)}{2l+1}[(a_{l+}-a_{l-})^2+(r_{l+}-r_{l-})^2] \quad (1)$$

$$A_0=(k^2/4\pi)\sigma^T=\sum[(l+1)a_{l+}+la_{l-}] \quad (2)$$

$$S=4k^2\frac{k}{\sqrt{s}}\frac{dA}{dt}\bigg|_{t=0}=\sum l(l+1)[(l+1)a_{l+}+la_{l-}]. \quad (3)$$

Here k is the CM momentum, a_{l+}, a_{l-}, r_{l+}, r_{l-} the imaginary and real parts of the partial waves.

The unitarity constraints are

$$u_l=a_{l+}-a_{l+}^2-r_{l+}^2\geqslant 0 \quad (4)$$

$$v_l=a_{l-}-a_{l-}^2-r_{l-}^2\geqslant 0. \quad (5)$$

Note that we have defined G here as $\frac{1}{2}$ of its definition in Sakmar (1981a, b) to make comparison of σ_{SF} and σ^T easier. We give below the definitions and formulae which will be needed in the application to the π^+–p problem. We refer the reader who is interested in the properties of the four classes and the derivations of the forms of the partial wave amplitudes in these classes to Sakmar (1981a, b).

We considered the following four classes.

I^+I^-. Both f_{l+} and f_{l-} are inelastic. This class was found to be empty.

I^+B^-. f_{l+} inelastic, f_{l-} elastic. This class can contribute only to saddle points, because the second derivative of the Lagrange function is positive in some cases. In this class we had two possibilities:

$$(1)\quad a_{l-}=1,\qquad a_{l+}=1-\frac{1}{2B}[\alpha+l(l+1)\beta],\qquad r_{l+}=r_{l-}=0 \quad (6)$$

$$(2)\quad a_{l-}=0,\qquad a_{l+}=-\frac{1}{2B}[\alpha+l(l+1)\beta],\qquad r_{l+}=r_{l-}=0. \quad (7)$$

Here α and β are Lagrange parameters to be determined by fitting the constraints. B is defined as

$$B\equiv 2l/(2l+1). \quad (8)$$

I^-B^+. f_{l+} elastic, f_{l-} inelastic. This class too can contribute only to saddle points, because the second derivative of the Lagrange function is positive in some cases. In this class we had two possibilities.

$$(1)\quad a_{l+}=1,\qquad a_{l-}=1-\frac{1}{2D}[\alpha+l(l+1)\beta],\qquad r_{l+}=r_{l-}=0 \quad (9)$$

(2) $a_{l+}=0, \qquad a_{l-}=-\frac{1}{2D}[\alpha+l(l+1)\beta], \qquad r_{l+}=r_{l-}=0.$ (10)

Here D is defined as

$$D \equiv 2(l+1)/(2l+1). \tag{11}$$

B^+B^-. Both partial waves are elastic. In this class we had five possibilities:

(1) $a_{l+}=a_{l-}=0, \qquad r_{l+}=r_{l-}=0$

(2) $a_{l+}=0, \qquad a_{l-}=1, \qquad r_{l+}=r_{l-}=0$

(3) $a_{l+}=1, \qquad a_{l-}=0, \qquad r_{l+}=r_{l-}=0$

(4) $a_{l+}=1, \qquad a_{l-}=1, \qquad r_{l+}=r_{l-}=0$

(5) $a_{l+}, a_{l-}, r_{l+}, r_{l-}\neq 0$. Their forms are given by equations (70)–(73) in Sakmar (1981a).

In addition to these formulae we had given also the contribution of different forms of the partial wave amplitudes in different classes to A_0, S and G. We shall not repeat those formulae here. They are given in Sakmar (1981a) by equations (34), (35), (40), (41), (50), (51), (56), (57) and (74)–(76).

The combination which enters into all formulae is

$$\alpha+l(l+1)\beta.$$

The unitarity and maximum condition impose on this combination lower and upper limits as functions of l. As we shall see, these limits will help us both in finding the solutions and in completing the search for all solutions.
Examples for these limits are:

$$\text{in } I^+B^-2 \qquad -\frac{4l}{2l+1}\leqslant\alpha+l(l+1)\beta\leqslant-\frac{2l(l+1)}{(2l+1)^2} \tag{12}$$

$$\text{in } I^-B^+2 \qquad -\frac{4(l+1)}{2l+1}\leqslant\alpha+l(l+1)\beta\leqslant\frac{2l(l+1)}{(2l+1)^2} \tag{13}$$

$$\text{in } B^+B^-5 \qquad -2\leqslant\alpha+l(l+1)\beta\leqslant-2/(2l+1). \tag{14}$$

These three regions are shown in figure 1.

3. Input

Our input is A_0 and S. We calculate these two quantities from phase shifts. As reference we used the UCRL report (1970). In addition to A_0 and S we also calculated G and E, which is in principle the elastic cross section.

$$E=(k^2/4\pi)\sigma^{\mathrm{el}}=\sum\,[(l+1)(a_{l+}^2+r_{l+}^2)+l(a_{l-}^2+r_{l-}^2)].$$

As can be seen from our tables different phase shift sets are not normalised to the same total cross section. For this reason, as a meaningful quantity we also calculated G/A_0. Finally, to get an idea about the elasticity we give E/A_0. Up to 1320 MeV there are fewer fits in our reference than at higher energies. We selected a large number of energies and calculated for different groups' fits at all energies, the quantities

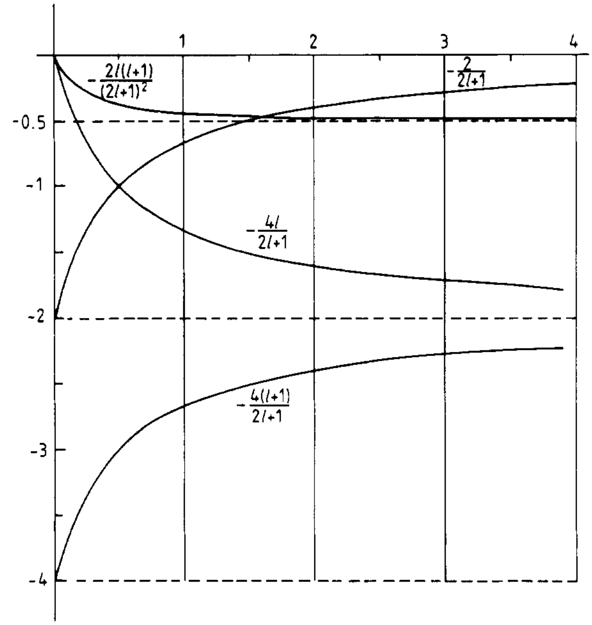

Figure 1. l-dependent functions which determine the boundaries of the quantity $\alpha + l(l+1)\beta$.

A_0, S, G, E, E/A_0 and G/A_0. Examples of the resulting data at one energy are given in table 1.

Table 1. Data calculated at one energy.

	π^+–p 1543 MeV					
	A_0	S	G	E	E/A_0	G/A_0
Roper via Moorehouse						
Berkeley Boone 21	0.8054	3.6361	0.0320	0.3744	0.4649	0.0397
Berkeley Path 1	0.7022	1.9310	0.0270	0.3837	0.5464	0.0384
Berkeley Path 2	0.7022	1.9310	0.0270	0.3837	0.5464	0.0384
Saclay	0.6931	2.5422	0.0350	0.3798	0.5480	0.0505
CERN Kirsopp	0.7572	1.6597	0.0180	0.4532	0.5985	0.0238
CERN Experimental	0.6881	1.5152	0.0289	0.3939	0.5724	0.0420
CERN Theoretical	0.7041	1.5281	0.0304	0.3928	0.5579	0.0432
Glasgow A (1540 MeV)	0.7333	1.9409	0.0133	0.4055	0.5529	0.0181
Glasgow B (1540 MeV)	0.7525	2.0267	0.0137	0.4238	0.5632	0.0182

4. Solutions

The solutions were searched for in the following way. First of all we look at the numerical values of A_0 and S. Then we insert for a_{l+} and a_{l-} their forms from different classes in (2) and (3). This in general restricts the l value because of the limits on $\alpha + l(l+1)\beta$ mentioned before. Then the limited number of possibilities are tried to fit A_0 and S. This gives the values of α and β. Finally these α and β are used to form $\alpha + l(l+1)\beta$ to check whether the unitarity and maximum conditions on this combination are satisfied. All cases which do not satisfy these conditions are eliminated.

We first want to demonstrate this process with an example. Later we will give the solutions we have found.

Consider π^+–p scattering at 1362 MeV. If we use Berkeley Boone 21 phase shifts we find

$$A_0 = 1.0221 \qquad S = 3.2851.$$

If a pair of partial wave amplitudes were in the class B^+B^-4 for example, then we would have

$$a_{l+} = 1 \qquad a_{l-} = 1$$

and their contribution to S from equation (3) would be for the lowest l value ($l = 1$)

$$1.2[2+1] = 6.$$

This already exceeds our S. Hence there can be no contribution from class B^+B^-4.

As another example consider a more likely class, for instance I^+B^-2. In this class

$$a_{l+} = -(1/2B)[\alpha + l(l+1)\beta] \qquad a_{l-} = 0.$$

Their contribution to S will be in equation (3)

$$-l(l+1)(l+1)[(2l+1)/4l][\alpha + l(l+1)\beta].$$

But the unitarity and maximum condition restrict $[\alpha + l(l+1)\beta]$ in I^+B^-2 by the relation (12). This means that

$$-[\alpha + l(l+1)] > 0.44 \qquad \text{for } l \geqslant 1.$$

Hence for $l = 1$ the contribution to S is greater than

$$2 \times 2 \times \tfrac{3}{4} \times 0.4444 = 1.32.$$

For $l = 2$ the contribution to S is greater than

$$3 \times 3 \times \tfrac{7}{4} \times 0.4747 = 7.4025.$$

But this is larger than the given value of S. This shows that from this class not more than one partial wave (e.g. $l = 1$) can contribute.

We now give some of the solutions we have found. With these solutions we calculate G and the solution which gives the largest G will be the one we are looking for. This G should be compared with the input A_0 rather than the G obtained from phase shifts since we are using a very limited input. Moreover, the G values found from phase shifts are not consistent among themselves. In the computer program the eight classes of partial waves I^+B^-1, I^+B^-2, I^-B^+1, I^-B^+2, B^+B^-2, B^+B^-3, B^+B^-4 and B^+B^-5 are called respectively 1, 2, 3, 4, 5, 6, 7 and 8. In addition the zero class is numbered 9.

Thus 2 4 8 9 9 means that in a five partial wave search ($l = 1, \ldots, 5$) the $l = 1$ wave is taken from class I^+B^-2, the $l = 2$ wave from class I^-B^+2 and the $l = 3$ wave from class B^+B^-5. The $l = 4$ and $l = 5$ waves are zero.

We did at most energies seven, and in some cases also eight wave searches. Actually, because of the restrictions imposed on the number of waves already discussed one does not need to go much higher.

Since the extrema depend critically on the input in the variational calculus, we searched for the solutions for different phase shift fits at the same energies. The fits we used are the following, together with the labels we used for them.

Roger via Moorehouse RVM Berkeley Boone 21 BB 21

Berkeley Path 1 BP 1 Saclay pion–nucleon phase shifts S

CERN Kirsopp CK CERN theoretical fit CTF

Glasgow solution A GSA.

We give below some of our solutions. The one we are looking for is the largest for a given group's fit. As discussed previously, because some Lagrange parameters are zero in some classes, we do not have control on certain second derivatives in such classes. These are the classes 1–4. In the remaining classes λ and μ are not zero and can be chosen to make second derivatives negative. These classes are 5–8. Hence the solutions which contain 1–4 are saddle points whereas the solutions with only 5–8 are maxima. Among the maxima we have to choose the largest to find the upper bound.

For a specified number of partial waves the computer program picks from nine different classes, partial waves with their characteristic forms in their respective classes; it then fits with these waves A_0 and S and calculates α and β which give the fit. Once α and β are found, it forms the numerical value of partial waves and checks it against the bounds imposed on partial waves by unitarity and maximum conditions. If the partial waves satisfy these conditions it writes out the values of α and β and also calculates G. If they do not, it rejects it. In table 2 we give a summary of the results. (NS) means no solution, (M) means maximum and (Sa) means saddle point.

Table 2. Summary of the results. RVM phase shifts go only up to 1320 MeV. At lower energies no solutions were found for RVM and CK. Above 1673 we checked only BP1 and CTF.

	RVM	BB21	BP1	S	CK	CTF	GSA
1320	NS	Sa	NS	NS	NS	NS	NS
1362		M, Sa	NS	NS	NS	NS	NS
1390		M, Sa	NS	NS	NS	NS	NS
1470		M, Sa	NS	NS	NS	NS	NS
1542		M, Sa	M, Sa	M, Sa	M, Sa	M, Sa	M, Sa
1673		M, Sa	M, Sa	Sa	Sa	M, Sa	Sa
1737			M, Sa			M, Sa	
1821			M, Sa			M, Sa	
1969			M, Sa			M, Sa	

We also list the form of the solutions as well as the value of G in cases for which solutions were found. Because of the large number of solutions we give only the largest maximum together with a saddle point. In a few cases there are no maxima but only saddle points (1673S, 1672SK, 1680GSA). In one case there is only one maximum (1672CTF). The existence or the goodness of the bound depends critically on the input values of A_0 and S rather than the energy. Thus at 1362, 1390 and 1470 MeV, of all the phase shift fits, only Berkeley Boone 21 has solutions. At large energies (1737, 1821, 1969 MeV) the number of solutions is very large.

	A_0		S	G	E		E/A_0	G/A_0
1320 BB21	1.1336		2.4038	0.1956	1.0505		0.9268	0.1725
Solutions		α		β		G		
1 899 999		2.8869		−0.8099		0.6360		Only saddle points
1362 BB21	1.0221		3.2851	0.0998	0.8495		0.8311	0.0976
8 899 999		−0.7004		−0.1752		0.8340		Maximum
4 999 998		−2.6891		0.0124		0.6885		Saddle point
1390 BB21	0.9744		3.0689	0.0489	0.7792		0.7996	0.0502
8 899 999		−0.7232		−0.1755		0.7980		
4 989 999		−2.3725		0.0364		0.6061		
1470 BB21	0.7650		2.4733	0.0125	0.5421		0.7087	0.0164
8 899 999		−1.0361		−0.1292		0.6605		
4 899 999		−1.2108		−0.1001		0.4114		
1543 BB21	0.8054		3.6361	0.0319	0.3744		0.4649	0.0397
8 889 999		−1.4109		−0.0468		0.7294		
4 899 999		−0.3990		−0.1993		0.5147		
1543 BP1	0.7022		1.9311	0.0270	0.3837		0.5464	0.0384
8 899 999		−0.9117		−0.1638		0.5902		
4 999 989		−1.8294		−0.0040		0.3297		
1543 S	0.6931		2.5422	0.0399	0.3798		0.5480	0.0575
8 899 999		−1.3075		−0.0769		0.6222		
4 899 999		−0.7318		−0.1728		0.3812		
1543 CK	0.7572		1.6597	0.0180	0.4532		0.5985	0.0237
8 899 999		−0.5727		−0.2330		0.5837		
4 888 888		−1.1999		−0.00002		0.3822		
1543 CTF	0.7041		1.5282	0.0304	0.3928		0.5579	0.0863
8 899 999		−0.6637		−0.2187		0.5525		
4 999 989		−1.8632		−0.0032		0.3307		
1540 GSA	0.7333		1.9409	0.0123	0.4055		0.5529	0.0180
8 899 999		−0.8180		−0.1812		0.6045		
4 999 998		−1.9296		−0.0012		0.3588		
1672 BB21	1.7599		8.0290	0.0845	0.7546		0.4287	0.0480
88 899 999		−0.7239		−0.1008		1.4009		
48 999 999		−1.9816		0.1472		1.1400		
1672 BP1	1.7912		9.0810	0.1686	0.7853		0.4384	0.0942
8 889 999		−0.8225		−0.0875		1.4626		
2 498 999		−0.4536		−0.0742		0.4412		
1673 S	1.6159		4.8731	0.0426	0.7655		0.4737	0.0264
1 499 999		2.0406		−0.4222		0.5872		Only saddle points
1 899 999		2.6303		−0.7170		0.9133		
1672 CK	1.6323		5.3859	0.1053	0.7769		0.4760	0.0645
1 899 999		2.6859		−0.7103		1.0746		Only saddle points
2 899 999		−0.3141		−0.2103		0.6765		
1672 CTF	1.3281		5.0386	0.1068	0.4960		0.3775	0.0804
8 899 999		−0.7734		−0.1250		1.0783		
4 899 999		−2.1680		0.1074		0.8823		
1680 GSA	1.6516		5.8639	0.0745	0.7989		0.4837	0.0451
1 499 999		2.3726		−0.5235		0.7820		Only saddle points

	A_0		S	G	E		E/A_0	G/A_0
Solutions		α		β		G		
1737 BP1	2.1666		9.8898	0.15045	1.0645		0.4913	0.0695
8 889 999		−0.4298		−0.1241		1.6229		
4 899 999		−2.6657		0.2962		1.4062		
1738 CTF	2.0317		8.5660	0.1545	1.0690		0.5261	0.0760
8 889 999		−0.4331		−0.1281		1.5103		
5 489 999		−0.3618		−0.1335		1.0105		
1821 BP1	3.0659		21.5281	0.2095	1.3195		0.4304	0.0684
8 888 999		−0.6892		−0.0640		2.4055		
9 289 999		−1.0117		−0.0575		1.3460		
1821 CTF	2.9602		18.5002	0.2615	1.5116		0.5106	0.0883
8 889 999		−0.5198		−0.0864		2.2642		
4 449 999		−2.7242		0.1817		1.1756		
1969 BP1	4.4637		34.4504	0.2934	2.0497		0.4592	0.0658
5 888 999		−0.0034		−0.0955		3.1762		
2 248 999		−0.4554		−0.0764		1.1846		
1968 CTF	4.3304		31.0333	0.2983	1.9260		0.4448	0.0689
5 888 999	0.1004		−0.1047		3.0085			
6 899 899		−0.2628		−0.0513		2.3318		

5. Discussion and conclusion

Since the phase shifts normally fit the differential cross sections and polarisations, a comparison of the values of G or perhaps more meaningfully G/A_0, which takes normalisation into account, reveals a great difference between different phase shift sets. Thus at 1320 MeV the largest and smallest values of G/A_0 differ by 25%, at 1443 MeV by 460% and at 1543 MeV by 180%. For S, the corresponding values are 35, 55 and 140 per cent. This demonstrates the need for the measurement of the rotation parameters or theoretical methods to choose the best fits. To demonstrate the application of the method in the most general case, that is when partial waves from different classes can contribute, we searched for both the maxima and saddle points. We have already discussed how the solutions are searched for by taking partial waves from different classes. This may seem like a tedious job. However, because A_0 and S are finite and, for the energies we have investigated, small numbers, the number of partial waves from all classes except B^+B^-5 is limited and in most cases only one or two. This is because the partial waves are larger than 0.44 for $l \geq 1$. Only in B^+B^-5 can the partial waves be very small and their number large. However the effect of increasing the number of partial waves from B^+B^-5 can be followed, be it by individual calculations or by computer programming. One finds that if the partial waves with terminal l values from B^+B^-5 do not satisfy unitarity and maximum conditions, then when the l values are increased, the tail end of partial waves will still not satisfy these conditions. When only the class B^+B^-5 contributes, from the forms of G, A_0 and S in this class one finds that the maximum of G is related to A_0 and S by $G = \frac{1}{2}[(2-\alpha)A_0 - \beta S]$. At some energies we could not find any solution at all. This is understandable, because in a variational problem in which the domain of the variables is restricted the maximum may lie outside this domain.

On the other hand, the existence and type of the solutions depend on the numerical values of A_0 and S. Therefore experimental values of these quantities are critical for the solutions. It is only for convenience and demonstration of the method that we constructed A_0 and S from phase shifts. In principle one should use the data directly. As a byproduct one finds of course, a great difference between various phase shifts.

When at a given energy several solutions exist we have to take the largest maximum. It is to be noted that whenever the partial waves are in classes 5–8, and especially in 8, G is larger. The reason for this is that in this class there is an ambiguity in the sign of the real parts coming from the square root. For this reason in equation (1) one has to choose the worst combination for $(r_{l+}-r_{l-})^2$, that is r_{l+} and r_{l-} are chosen with opposite signs and this makes this term large. Actually the forms of the partial waves in B^+B^-5 are found such that r_{l+} and r_{l-} have opposite signs. At inelastic energies G can even be larger than E. However, we should realise that the input is only A_0 and S, and in the absence of the variational calculus the upper bound of G is A_0. Thus in all cases there is an improvement on information on G.

We are in the process of applying this analysis to other spin-$\frac{1}{2}$–spin-0 processes which will be completed shortly.

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